The Nordhaus DICE-model (Managing the global commons)
DEA-CCAT/Jesper Gundermann & Peter Laut 1997 rev 27.06.99

$nmax := 60$ sixty timesteps á 10 years

Population model:

$LL0 := 3369$ 1965 world population, millions

$GL0 := .223$ growth rate population per decade

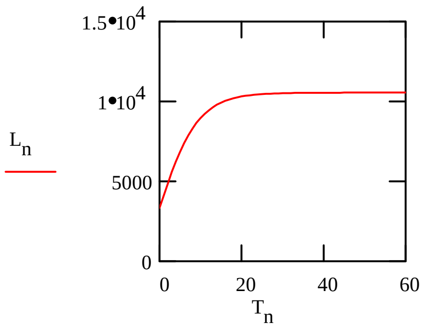
$DLAB := .195$ decline rate of population growth per decade

$$GL(t) := \frac{GL0}{DLAB} \cdot (1 - \exp(-DLAB \cdot t))$$

$$Lf(t) := LL0 \cdot \exp(GL(t))$$

$n := 0 .. nmax$ $\quad T_n := n$

$$L := \overrightarrow{Lf(T)}$$



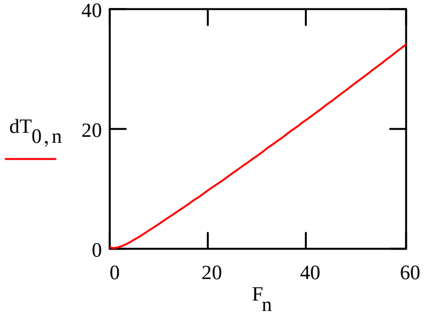
Construct forcing-climate relation $\quad F_n := n$

$C1 := 0.226$ $\quad C3 := 0.440$ $\quad C4 := 0.02$ $\quad LAM := 1.41$

$$dTf(F) := \left| \begin{array}{l} dT^{\langle 0\rangle} \leftarrow \begin{bmatrix} .2 \\ .1 \end{bmatrix} \\ \text{for } n \in 1 .. nmax + 1 \\ \quad dT^{\langle n\rangle} \leftarrow \begin{bmatrix} 1 - C1 \cdot LAM - C1 \cdot C3 & C1 \cdot C3 \\ C4 & 1 - C4 \end{bmatrix} \cdot dT^{\langle n-1\rangle} + \begin{bmatrix} C1 \cdot F_{n-1} \\ 0 \end{bmatrix} \\ dT \end{array} \right.$$

$dT := dTf(F)$ $\qquad$ $dT0 := \left(dTf(F \cdot 0)^T\right)^{\langle 0\rangle}$

$n := 0 .. nmax$



Carbon model, from Emissions to Carbon of atmosphere:

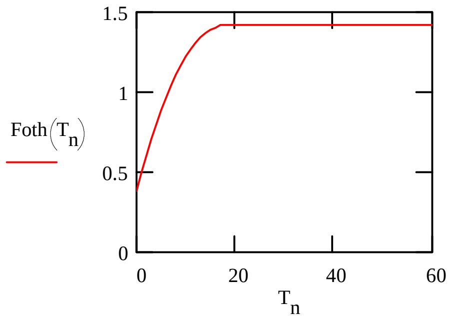
$ATRET := 0.64$ $\qquad$ $DELTAM := 0.0833$ $\qquad$ $M0 := 677$

$$Mf(E) := \left| \begin{array}{l} M_0 \leftarrow M0 \\ \text{for } n \in 1 .. nmax + 1 \\ \quad M_n \leftarrow 590 + ATRET \cdot E_{n-1} + (1 - DELTAM) \cdot \left(M_{n-1} - 590\right) \\ M \end{array} \right.$$

$E_n := n$ $dM := Mf(E)$ $dM0 := Mf(E \cdot 0)$

Forcing, other GHG's:

$$Foth(t) := if\left[t<17, 0.2604 + 0.125\cdot(t+1) - 0.0034\cdot(t+1)^2, 1.42\right]$$



$$Ff(Mc,t) := 4.1\cdot\ln\left(\frac{Mc}{590}\right)\cdot\frac{1}{\ln(2)} + Foth(t)$$

Investments and Capital stock:

$K0 := 16.03$ $DK := 0.1$ $DKfac := (1 - DK)^{10}$

$$Kf(I) := \left| \begin{array}{l} K_0 \leftarrow K0 \\ \text{for } n \in 1 .. nmax + 1 \\ \quad K_n \leftarrow DKfac\cdot K_{n-1} + 10\cdot I_{n-1} \\ K \end{array} \right.$$

$n := 0 .. nmax$

$I_n := n$ $dK := Kf(I)$ $dK0 := Kf(I\cdot 0)$

Note, the relation between K and I is: $I = MKtoI\cdot K$ where:

$n := 1..nmax$ $\qquad MKtoI_{n-1,n-1} := -\frac{DKfac}{10}$ $\qquad MKtoI_{n-1,n} := \frac{1}{10}$

$MKtoI_{nmax,nmax} := \frac{1 - DKfac}{10}$ (assuming Kn+1 = Kn)

Subroutines for calculating response from ramp-response R and input, F:

```
dp2(A) := | k ← last(A)
          | for kk ∈ 0.. k − 1
          |   res_kk ← A_(kk+1) − A_kk
          | res
```

```
dp(A) := | k ← last(A)
         | for kk ∈ 0.. k −
         |   res_kk ← A_(kk+1) -
         | res_k ← 0
         | res

dm(A) := | k ← last(A)
         | res_0 ← A_0
         | for kk ∈ 1.. k
         |   res_kk ← A_kk −
         | res

RES(F, R, dt) := | d2F ← dm(dp(F))·(1/dt)
                 | for n ∈ 0.. last(F)
                 |   res_n ← Σ_(k=0)^(n) d2F_(n−k)·R_k
                 | res
```

Subroutines for calculating response-matrices:

```
u(n, k) := | for kk ∈ 0.. n
           |   uu_kk ← 0
           | uu_k ← 1
           | uu

MT(SC, dt, n) := | for k ∈ 1.. n
                 |   | uu ← u(n, k)
                 |   | mc^<k> ← RES(uu, SC, dt)
                 | mc_(0,0) ← 1
                 | mc
```

Check Ramp responses:

Check Ramp responses:

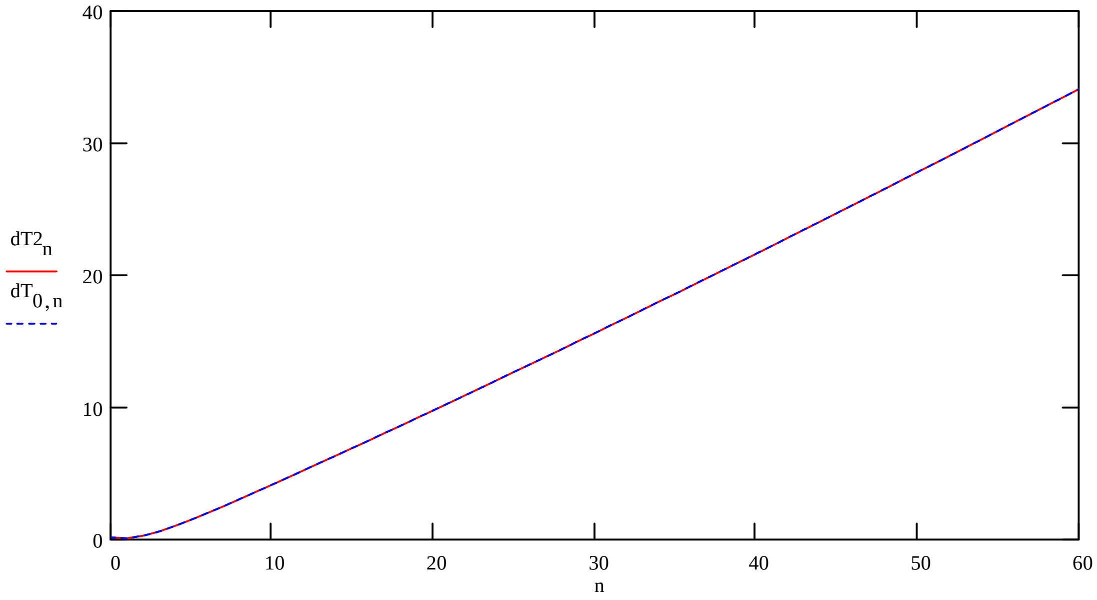
Temperature:

$n := 0 .. nmax + 1$

$RT_n := \left(dT_{0,n} - dT0_n\right) \cdot 10$ $\quad$ $Mtemp := submatrix(MT(RT, 10, nmax + 1), 1, nmax + 1, 1, nmax + 1)$

$n := 0 .. nmax$

$dT2_n := (Mtemp \cdot F)_n + dT0_n$ $\quad$ $rows(Mtemp) = 61$ $\quad$ $rows(F) = 61$



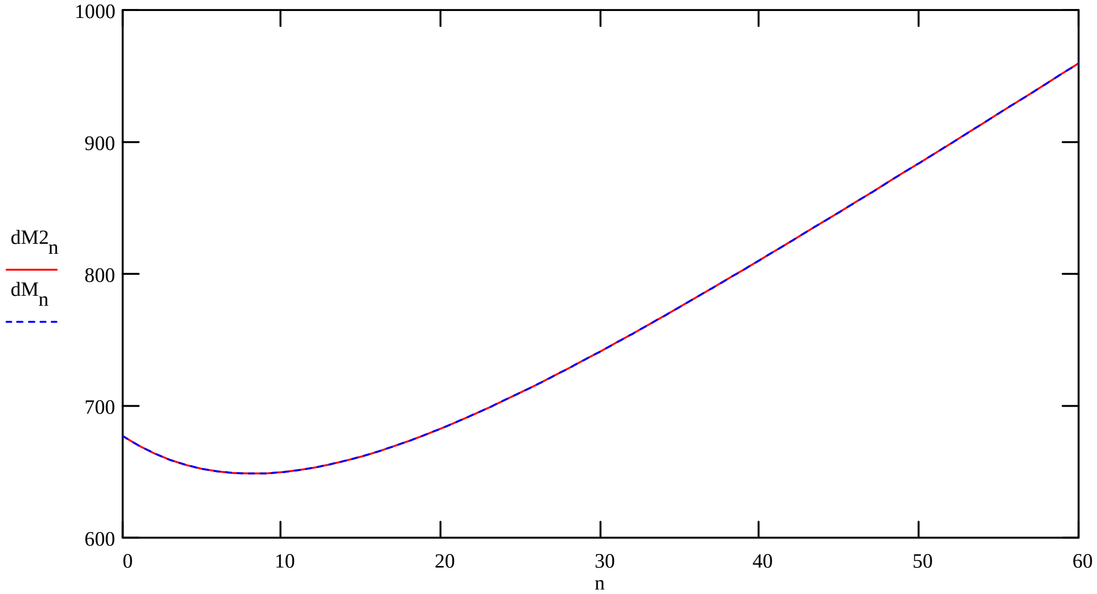
Carbon:

$n := 0 .. nmax + 1$

$RC_n := \left(dM_n - dM0_n\right) \cdot 10$ $\quad$ $Mcarb := submatrix(MT(RC, 10, nmax + 1), 1, nmax + 1, 1, nmax + 1)$

$n := 0 .. nmax$

$dM2_n := dM0_n + (Mcarb \cdot E)_n$



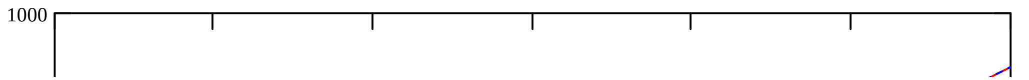
Capital stock:

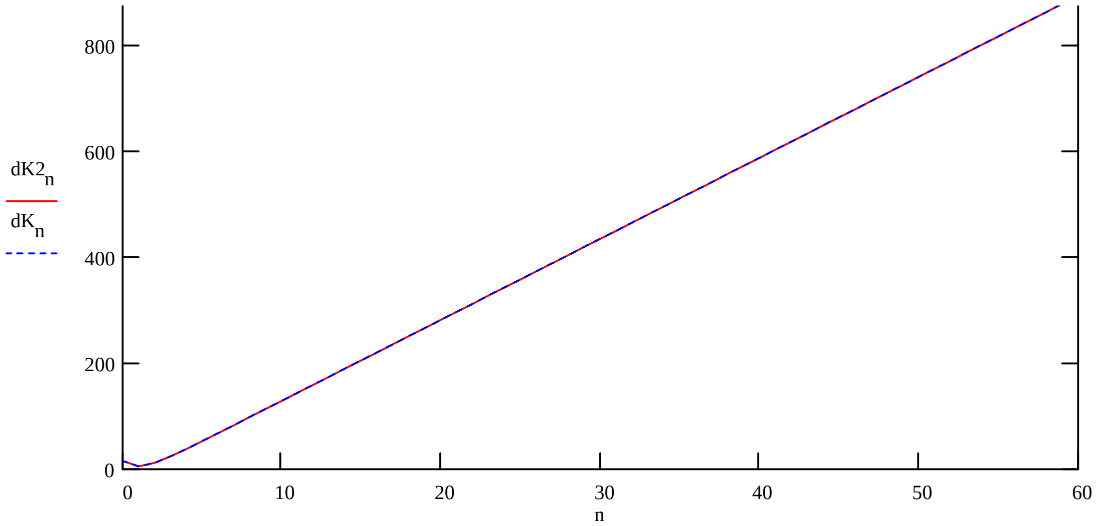
$n := 0 .. nmax + 1$

$RK_n := \left(dK_n - dK0_n\right) \cdot 10$ $\quad$ $Mcap := submatrix(MT(RK, 10, nmax + 1), 1, nmax + 1, 1, nmax + 1)$

$n := 0 .. nmax$

$dK2_n := dK0_n + (Mcap \cdot I)_n$ $\quad$ $rows(dK2) = 61 \quad rows(Mcap) = 61$





Discounting factors (r=rate of pure time preference, set below):

$n := 0 .. nmax$ $\quad EXP_n := (1 + r)^{-n \cdot 10}$ $\quad EXPINC_n := (1 + r)^{-(nmax - n) \cdot 10}$

Coefficients for steady-state price relations (related to Nordhaus "transversality relations")

$Rtemp := (Mtemp \cdot EXPINC)_{nmax}$ $\quad Rtemp = 0.299$

$Rcap := (Mcap \cdot EXPINC)_{nmax}$ $\quad Rcap = 10.048$

$Rcarb := (Mcarb \cdot EXPINC)_{nmax}$ $\quad Rcarb = 1.498$

$RKtoI := (MKtoI \cdot EXPINC)_{nmax}$ $\quad RKtoI = 0.065$

World economic output, and C-emissions:

$b1 := 0.0686 \cdot 1$ $\quad b2 := 2.887$ $\quad A1 := 0.0133 \cdot 1$

$DELA := 0.11$ $\quad GSIGMA := -.1168$

$GSIG(t) := \left(\frac{GSIGMA}{DELA}\right)\cdot(1 - \exp(-DELA\cdot t))$ $SIG0 := 0.519$

$\sigma(t) := SIG0\cdot\exp(GSIG(t))$

$\Omega(\mu, dT) := \frac{1 - b1\cdot\mu^{b2}}{1 + \frac{A1}{9}\cdot dT^2}$ Nordhaus reduction factor for economic output due to mitigation (μ) and Climate Change (dT)

$\mu2(pY, pE, dT, t) := \left[\frac{pE}{pY}\cdot\left(1 + \frac{A1}{9}\cdot dT^2\right)\cdot\frac{1}{b1\cdot b2}\cdot 10\cdot\sigma(t)\right]^{\frac{1}{b2-1}}$ Controle parameter (Fraction of BaU emissions avoided)

$\mu(pY, pE, dT, t) := if(\mu2(pY, pE, dT, t) \le 1, \mu2(pY, pE, dT, t), 1)$

$a0 := 0.00963$ $GA0 := 0.15$ $\mu(pY, pE, dT, t) := 0$ ▪   use this expression to run reference scenario (No control)

$GA(t) := \frac{GA0}{DELA}\cdot(1 - \exp(-DELA\cdot t))$ $AL(t) := a0\cdot\exp(GA(t))$

$\gamma := 0.25$ $Q0(L, K, t) := AL(t)\cdot L^{1-\gamma}\cdot K^{\gamma}$ $b1 = 0.069$

Dealing with the production process:

The starting point is ordinary Cobb-Douglas: $Y(K, L, \mu, dT) = \Omega\cdot AL\cdot K^{\gamma}\cdot L^{1-\gamma}$

Emissions proportional to output! $E(K, L, \mu, dT) = (1 - \mu)\cdot 10\cdot\sigma\cdot AL\cdot K^{\gamma}\cdot L^{1-\gamma}$

This is changed to sensible variables (activities or prices) by:

$Y2(E, K, L, dT) = Y$ $pE = pY\cdot\frac{\delta Y2}{\delta E} = pY\cdot\frac{\delta Y}{\delta\mu}\cdot\left(\frac{\delta E}{\delta\mu}\right)^{-1}$ $pK = pY\cdot\frac{\delta Y}{\delta K}$

from which are found: $\mu(pE, pY, dT)$ $E(pY, pE, dT, L, pK)$ $K(pY, pE, dT, L, pK)$ and $Y(pY, pE, dT, L, pK)$

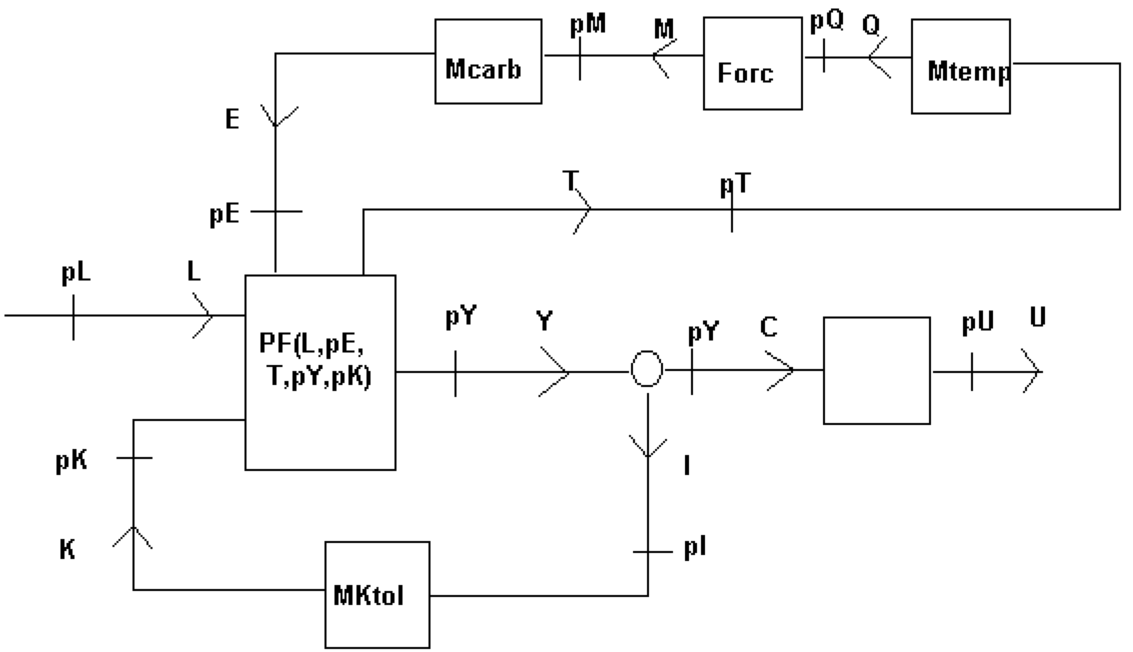
The Production Function PF then satisfies: $E = -\frac{\delta PF}{dpE}$ $pL = \frac{\delta PF}{\delta L}$ $pT = \frac{-\delta PF}{\delta dT}$ $Y = \frac{\delta PF}{\delta pY}$ $K = \frac{-\delta PF}{\delta pK}$

(There is a minus sign, whenever the variables in the diagram below looks like -|--<-- (k-orientation))

System overview, and ordering of variables: $PF = pY \cdot Y - pE \cdot E - pK \cdot K$ Production Function

Note PF is sum over the branches that have price (indicated by --|--) as independent variable, with a minus when the branch has K-orientation.

When a consistent solution has been found, it may be shown, that this maximises the production-function for the whole diagram (which is the utility pU*U, where pU=1)



$$Kf(pY, pE, dT, L, pK, t) := L\cdot\left[\frac{pK}{\gamma\cdot AL(t)\cdot(\Omega(\mu(pY, pE, dT, t), dT))\cdot pY}\right]^{\frac{1}{\gamma-1}}$$

$$Ef(pY, pE, dT, L, pK, t) := (10\cdot\sigma(t)\cdot(1-\mu(pY, pE, dT, t)))\cdot Q0(L, Kf(pY, pE, dT, L, pK, t), t)$$

$$Yf(pY, pE, dT, L, pK, t) := \Omega(\mu(pY, pE, dT, t), dT)\cdot Q0(L, Kf(pY, pE, dT, L, pK, t), t)$$

$$PF(pY, pE, dT, L, pK, t) := pY\cdot Yf(pY, pE, dT, L, pK, t) - (pE\cdot Ef(pY, pE, dT, L, pK, t)) - pK\cdot Kf(pY, pE, dT, L, pK, t)$$

$$pTf(pY, pE, dT, L, pK, t) := -\left(\frac{d}{d\,dT}PF(pY, pE, dT, L, pK, t)\right) + \text{if}\left[dT < TMAX, 0, \left(\frac{dT - TMAX}{100 + t}\right)\cdot ALFAT\cdot 100\right]$$

$$pLf(pY, pE, dT, L, pK, t) := \frac{d}{dL}PF(pY, pE, dT, L, pK, t)$$

$pdI^T\cdot I = pdI^T\cdot MKtoI\cdot K$ ie $pdK^T = pdI^T\cdot MKtoI$ $pdK = MKtoI^T\cdot pdI$

$MKtoItrans := MKtoI^T$ $MKtoItrans_{nmax, nmax} := MKtoItrans_{nmax-1, nmax-1}$

$$dK0 := submatrix(dK0, 0, nmax, 0, 0)$$

$$dT0 := submatrix(dT0, 0, nmax, 0, 0)$$

$$dM0 := submatrix(dM0, 0, nmax, 0, 0)$$

Initial values:

$Zer := EXP\cdot 0$ $dT := Zer + \frac{T}{400}$ $pE := Zer + 10^{-3}$ $K0 = 16.03$

$I := \overrightarrow{.1\cdot(1 - DKfac)\cdot K0 + Zer}$ $I_0 = 1.044$

$K := Mcap \cdot I + dK0$

$pY := \overrightarrow{\frac{L}{.55 \cdot (8.519 - I)}}$

$pdY := \overrightarrow{pY \cdot EXP}$

$my := \overrightarrow{\mu(pY, pE, dT, T)}$

$pdK := MKtoItrans \cdot pdY$ $\qquad$ $pK := \overrightarrow{EXP^{-1} \cdot pdK}$

$pK_0 := \gamma \cdot AL(0) \cdot \left(\Omega\left(\mu\left(pY_0, pE_0, dT_0, 0\right), dT_0\right)\right) \cdot \left(\frac{L_0}{K0}\right)^{1-\gamma} \cdot pY_0$

$C := \overrightarrow{Yf(pY, pE, dT, L, pK, T) - I}$

Start of iteration Loop: disable (toggle) during first run

$(pY \ \ pE \ \ dT \ \ L \ \ pK \ \ T \ \ C) := if(re = 1, READPRN("Nordhau4.prn"), (pY \ \ pE \ \ dT \ \ L \ \ pK \ \ T \ \ C))$

Dependent variables, Production-process:

$Y := \overrightarrow{Yf(pY, pE, dT, L, pK, T)}$

$E := \overrightarrow{Ef(pY, pE, dT, L, pK, T)}$

$pT := \overrightarrow{pTf(pY, pE, dT, L, pK, T)}$

$K := \overrightarrow{Kf(pY, pE, dT, L, pK, T)}$ $\qquad$ $K_0 = 16.03$ $\qquad$ $K0 = 16.03$

Investments and consumption:

$I := MKtoI \cdot K$

$C := Y - I$

Marginal utility of consumption (from U=L*ln(C/L)/.55):

$$pY := \overrightarrow{\frac{L}{.55 \cdot C} \cdot w + (1 - w) \cdot pY} \qquad pdY := \overrightarrow{EXP \cdot pY}$$

Shadow price, capital:

$$pdK := MKtoItrans \cdot pdY \qquad pK := \overrightarrow{EXP^{-1} \cdot pdK}$$

$$pK_0 := \gamma \cdot AL(0) \cdot \left(\Omega\left(\mu\left(pY_0, pE_0, dT_0, 0\right), dT_0\right)\right) \cdot \left(\frac{L_0}{K0}\right)^{1-\gamma} \cdot pY_0$$ Special case

Carbon in atmosphere:

$$M := dM0 + Mcarb \cdot E$$

Forcing:

$$Q := \overrightarrow{Ff(M, T)}$$

Temperature increase:

$$dT := (dT0 + Mtemp \cdot Q) \cdot w + (1 - w) \cdot dT$$

Shadow-price, Forcing:

$$pdQ := reverse\left(Mtemp \cdot reverse\left(\overrightarrow{\left(pT - pT_{nmax}\right) \cdot EXP}\right)\right) + Rtemp \cdot pT_{nmax} \cdot EXP$$

$$pQ := \overrightarrow{pdQ \cdot EXP^{-1}}$$

Shadow-price of Carbon in atmosphere:

$$pM := \frac{4.1}{\ln(2)} \cdot \overrightarrow{\frac{pdQ}{M \cdot EXP}}$$

Shadow-price of emissions:

$$pdE := reverse\left(Mcarb \cdot reverse\left(\overrightarrow{\left(pM - pM_{nmax}\right) \cdot EXP}\right)\right) + Rcarb \cdot pM_{nmax} \cdot EXP$$

$$pE := \overrightarrow{\left(pdE \cdot EXP^{-1}\right) \cdot w + (1 - w) \cdot (pE)}$$

$pE := \left(pdE \cdot EXP^{\ \ }\right) \cdot w + (1 - w) \cdot (pE)$

Some interesting functions:

$P := M \cdot \frac{280}{590}$ CO2-concentration ppmv

$my := \overrightarrow{\mu(pY, pE, dT, T)}$ Mitigation parameter (fraction of potential emissions avoided)

$Cfrac := \frac{\vec{C}}{Y}$ Fraction of net output consumed

$pCarb := \overrightarrow{pE \cdot pY^{-1} \cdot 10^{4}}$ Shadow-price emissions, $ 1989 per t C

$PerCapConsum := \overrightarrow{\frac{C}{L} \cdot 1000}$ Per Capita consumption, 1000 $1989

$Damfrac := \overrightarrow{A1 \cdot \frac{1}{9} \cdot dT^{2}}$ Damage, fraction of GDP

$Mitfrac := \overrightarrow{b1 \cdot my^{b2}}$ Mitigation costs, fraction of GDP

Solution from last iteration step: $n := 1 .. nmax - 1$ $DUM_{n} := 1$ $DUM_{nmax} := 0$

$(pYo\ \ pEo\ \ dTo\ \ Lo\ \ pKo\ \ To\ \ Co) := if(re = 1, READPRN(\text{"Nordhau4.prn"}), (Re(pY)\ \ Re(pE)\ \ Re(dT)\ \ L\ \ Re(pK)\ \ Re(T)\ \ Re(C)))$

$$err := \overrightarrow{\left[\left(\frac{pYo - pY}{pY}\right)^{2} + \left(\frac{pEo - pE}{pE}\right)^{2} + \left(\frac{dTo - dT}{dT}\right)^{2} + \left(\frac{Lo - L}{L}\right)^{2} + \left(\frac{pKo - pK}{pK}\right)^{2} + \left(\frac{Co - C}{C}\right)^{2}\right] \cdot DUM}$$

$CC := \overrightarrow{\ln\left(\frac{C}{L}\right) \cdot EXP}$ $CC \cdot L = -1.152 \cdot 10^{5}$ $A1 = 0.013$

$err1 := \overrightarrow{\left(\frac{pKo - pK}{pK}\right)^{2}}$ $err1 \cdot err1 = 0$ $n := 0 .. nmax$ $DUM \cdot E = 1.38 \cdot 10^{4}$

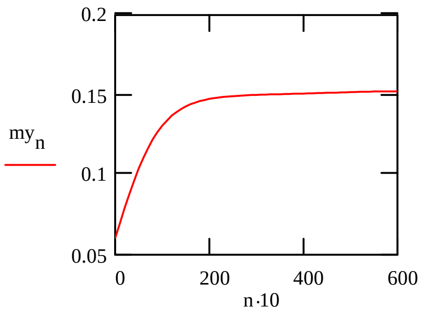
Store solution, for next iteration: PRNPRECISION := 6

WRITEPRN("Nordhau4.prn") := (Re(pY) Re(pE) Re(dT) L Re(pK) Re(T) Re(C))

After first run, enable READ above, put cursor in highlighted equation, and press F9 until err*err=0

Note n=0 is 1965, in the plots below

$r \equiv .03$ $\quad err \cdot err = 0$ $\quad g \equiv 1$ $\quad w \equiv .5$ $\quad re \equiv 1$ $\quad eps \equiv .0$ $\quad dT_{nmax} = 6.352$



$Y_0 = 8.52$ $\quad A1 \cdot \left(dT_{nmax} \cdot 3^{-1}\right)^2 = 0.06$

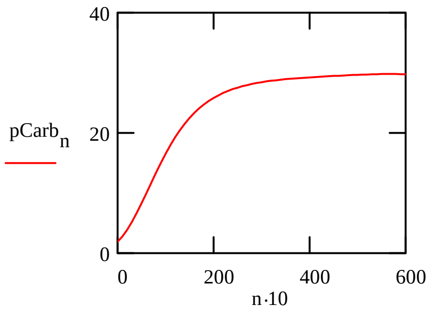
$I_0 = 1.872$ $\quad m := 0$ $\quad K_m = 16.03$

$\frac{pY_{20}}{pY_1} \cdot (1 + r)^{-200} = 5.874 \cdot 10^{-4}$ $\quad b1 \cdot \left(my_{nmax}\right)^{b2} = 2.989 \cdot 10^{-4}$

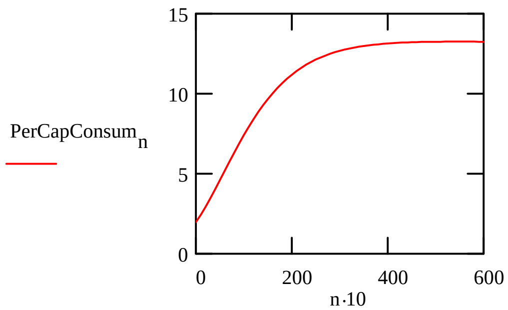
TMAX ≡ 1.5 ALFAT ≡ 10000·0

pK2 =

Emission shadow price ($ 1989 per t C)

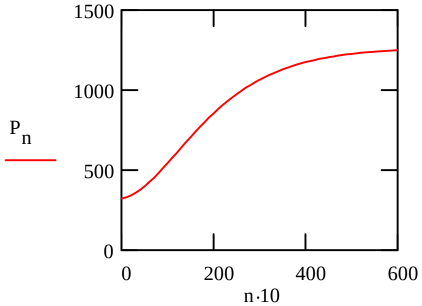


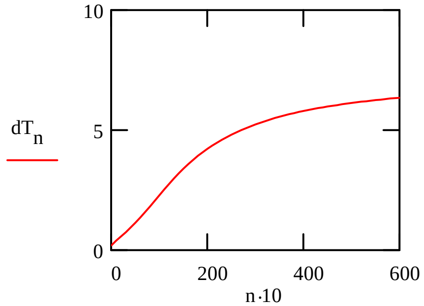
Per Capita Consumption
(1000 $/year)



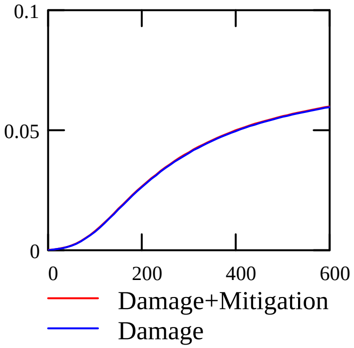
CO2 concentration ppmv

Temperature increase





Damage and mitigation costs, fraction of GDP



Note: it does'nt "pay-off" to use
any sizable fraction on mitigation!
(due to discounting the future)

pK =

	0
0	122.429
1	97.706
2	79.007
3	65.147
4	54.794
5	46.944
6	40.895
7	36.164
8	32.412
9	29.4
10	26.956
11	24.954
12	23.299
13	21.92
14	20.764